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Note

Flux-Vector Splitting for the Euler Equations for Real Gases

The flux-vector splitting for the Euler equations of compressible flow with the ideal-gas law used as equation of state was derived by van Leer [1] to approximate the hyperbolic system of conservation laws with so-called upwind differences. In this note we present the extension of the flux-vector splitting for the Euler equations of compressible flow with an arbitrary equation of state.

The three-dimensional Euler equations of compressible flow can be written in vector notation in Cartesian coordinates as

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{F}(\mathbf{U})}{\partial x}+\frac{\partial \mathbf{G}(\mathbf{U})}{\partial y}+\frac{\partial \mathbf{H}(\mathbf{U})}{\partial z}=0, \tag{1}$$

where the state vector $\mathbf{U}$ and the flux vectors $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$ have the form

$$\mathbf{U}=\begin{pmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ e\end{pmatrix}, \qquad \mathbf{F}(\mathbf{U})=\begin{pmatrix}\rho u\\ \rho u^2+p\\ \rho vu\\ \rho wu\\ (e+p)u\end{pmatrix},$$

$$\mathbf{G}(\mathbf{U})=\begin{pmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ \rho wv\\ (e+p)\,v\end{pmatrix}, \qquad \mathbf{H}(\mathbf{U})=\begin{pmatrix}\rho w\\ \rho uw\\ \rho vw\\ \rho w^2+p\\ (e+p)\,w\end{pmatrix}$$

Here ρ, u, v, w, e, and p are the density, the x-, y-, and z-component of the velocity, the total energy per volume and the pressure, respectively. The total energy e is given by $e=\frac{1}{2}\rho(u^2+v^2+w^2)+\varepsilon$, where ε is the internal energy per volume.

Using the definition of the speed of sound

$$c^2\equiv\Gamma_1\frac{p}{\rho}, \qquad \text{where} \quad \Gamma_1\equiv\left(\frac{d\ln p}{d\ln\rho}\right)_{\text{ad}}, \tag{2}$$

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and the quantity

$$\tilde{\Gamma}_3 \equiv \frac{p}{\varepsilon} + 1, \tag{3}$$

which for an equation of state of the form $p = \varepsilon f(\rho)$ is equal to the usual adiabatic index $\Gamma_3 \equiv (d \ln T/d \ln \rho)_{\text{ad}} + 1$, the flux vectors can be rewritten as

$$\mathbf{F}(\rho, c, M_x, M_y, M_z) = \begin{pmatrix} \rho c M_x \\ \rho c^2 (M_x^2 + 1/\Gamma_1) \\ \rho c^2 M_x M_y \\ \rho c^2 M_x M_z) \\ \rho c^3 M_x \{\frac{1}{2}(M_x^2 + M_y^2 + M_z^2) + \tilde{\Gamma}_3/[\Gamma_1(\tilde{\Gamma}_3 - 1)]\} \end{pmatrix}, \tag{4}$$

$$\mathbf{G}(\rho, c, M_x, M_y, M_z) = \begin{pmatrix} \rho c M_y \\ \rho c^2 M_y M_x \\ \rho c^2 (M_y^2 + 1/\Gamma_1) \\ \rho c^2 M_y M_z) \\ \rho c^3 M_y \{\frac{1}{2}(M_y^2 + M_x^2 + M_z^2) + \tilde{\Gamma}_3/[\Gamma_1(\tilde{\Gamma}_3 - 1)]\} \end{pmatrix}, \tag{5}$$

$$\mathbf{H}(\rho, c, M_x, M_y, M_z) = \begin{pmatrix} \rho c M_z \\ \rho c^2 M_z M_x \\ \rho c^2 M_z M_y \\ \rho c^2 (M_z^2 + 1/\Gamma_1) \\ \rho c^3 M_z \{\frac{1}{2}(M_z^2 + M_x^2 + M_y^2) + \tilde{\Gamma}_3/[\Gamma_1(\tilde{\Gamma}_3 - 1)]\} \end{pmatrix}. \tag{6}$$

The Mach numbers used in Eqs. (4)–(6) are defined as $M_x = u/c$, $M_y = v/c$, and $M_z = w/c$, respectively.

Given the flux-vectors in this modified form, it is straightforward to extend the flux-vector splitting of van Leer [1], which applies to ideal gases only, to the case of real gases. The resulting split fluxes for $|M| \leqslant 1$ read

$$\mathbf{F}^{\pm} = \begin{pmatrix} \pm\frac{1}{4}\rho c(\pm M_x + 1)^2 \\ \pm\frac{1}{4}\rho c^2(\pm M_x + 1)^2[(\Gamma_1 - 1)\, M_x \pm 2]/\Gamma_1 \\ \pm\frac{1}{4}\rho c^2(\pm M_x + 1)^2 M_y \\ \pm\frac{1}{4}\rho c^2(\pm M_x + 1)^2 M_z \\ \pm\frac{1}{4}\rho c^3(\pm M_x + 1)^2\{\alpha[(\Gamma_1 - 1)\, M_x \pm 2]^2 + \frac{1}{2}(M_y^2 + M_z^2)\} \end{pmatrix} \tag{7}$$

$$\mathbf{G}^{\pm} = \begin{pmatrix} \pm\frac{1}{4}\rho c(\pm M_y + 1)^2 \\ \pm\frac{1}{4}\rho c^2(\pm M_y + 1)^2 M_x \\ \pm\frac{1}{4}\rho c^2(\pm M_y + 1)^2[(\Gamma_1 - 1)\, M_y \pm 2)]/\Gamma_1 \\ \pm\frac{1}{4}\rho c^2(\pm M_y + 1)^2\, M_z \\ \pm\frac{1}{4}\rho c^3(\pm M_y + 1)^2\{\alpha[(\Gamma_1 - 1)\, M_y \pm 2]^2 + \frac{1}{2}(M_x^2 + M_z^2)\} \end{pmatrix}, \tag{8}$$

$$\mathbf{H}^{\pm}=\begin{pmatrix} \pm\frac{1}{4}\rho c(\pm M_z+1)^2 \\ \pm\frac{1}{4}\rho c^2(\pm M_z+1)^2 M_x \\ \pm\frac{1}{4}\rho c^2(\pm M_z+1)^2 M_y \\ \pm\frac{1}{4}\rho c^2(\pm M_z+1)^2[(\Gamma_1-1)M_z\pm 2)]/\Gamma_1 \\ \pm\frac{1}{4}\rho c^3(\pm M_z+1)^2\{\alpha[(\Gamma_1-1)M_z\pm 2]^2+\frac{1}{2}(M_x^2+M_y^2)\} \end{pmatrix}, \tag{9}$$

where α is defined as

$$\alpha\equiv\frac{1-\alpha_c}{2(\Gamma_1^2-1)} \quad \text{with} \quad \alpha_c=\frac{2(\tilde{\Gamma}_3-\Gamma_1)}{\Gamma_1(\Gamma_1+1)(\tilde{\Gamma}_3-1)}. \tag{10}$$

Note that for an ideal gas ($\gamma\equiv\Gamma_1=\tilde{\Gamma}_3$) the expressions for the split fluxes given above reduce to those obtained by van Leer [1]. For the one-dimensional case the same formulas have been independently derived by Montagne [2].

The split fluxes presented here do not exactly satisfy all constraints (1)–(7) imposed by van Leer [1] in the ideal gas case: Constraints (1) and (6), which van Leer considered crucial, and constraint (5) are only approximately satisfied. For real gases the coefficient appearing in the relation, which gives the split energy flux as a function of all other split fluxes, is no longer constant, but depends on the adiabatic indices Γ_1 and $\tilde{\Gamma}_3$ and therefore on the flow variables. Consequently a vanishing eigenvalue of $d\mathbf{F}^{\pm}/d\mathbf{U}$ for $|M|\leqslant 1$ (constraint (6)) and the continuity of the eigenvalues at $|M|=1$ (constraint (5)) is not ensured. In addition van Leer's constraint (1)—namely that the total flux is the sum of two split fluxes—is no longer exactly satisfied for the energy flux (component No. 5 of the flux vectors **F**, **G**, and **H**).

However, if Γ_1 and $\tilde{\Gamma}_3$ differ only by a few percent and are both slowly varying functions of the flow variables, the violation of contraints 1, 5, and 6 will have only little effect on numerical solutions. This is indeed confirmed by Arnett, Fryxell, and Müller [3], who applied the proposed generalized fluxes in simulations of thermonuclear burning fronts in degenerate stars, where the equation of state contains contributions due to ideal Boltzmann gases, radiation, and degenerate electrons (i.e., $\frac{4}{3}\leqslant\Gamma\leqslant\frac{5}{3}$). Concerning the violation of constraint (1) for the energy flux one can easily show after a little algebra that the size of the error is given by α_c (see Eq. (10)). For an equation of state consisting of a mixture of a $\Gamma=\frac{4}{3}$ and of a $\Gamma=\frac{5}{3}$ component, which is quite important in astrophysics, the resulting error is small: $\tilde{\Gamma}_3-\Gamma_1\leqslant 0.046$ and $\alpha_c\leqslant 0.042$. Test calculations of shock tube flows convincingly demonstrate that the generalized flux-splitter is capable of handling that equation of state accurately (Arnett, Fryxell, and Müller [3]).

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Ewald Müller

Max-Planck-Institut für Physik und Astrophysik,
Institut für Astrophysik,
Karl-Schwarzschild-Str. 1,
D-8046 Garching b. München,
Federal Republic of Germany